Question 1

On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

(a) $G'(5) = -24.588$ (or -24.587)

The rate at which gravel is arriving is decreasing by 24.588 (or 24.587) tons per hour per hour at time $t = 5$ hours.

$2 : \begin{cases} 1 : G'(5) \\ 1 : \text{interpretation with units} \end{cases}$

(b) $\int_0^8 G(t)\,dt = 825.551$ tons

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) $G(5) = 98.140764 < 100$

At time $t = 5$, the rate at which unprocessed gravel is arriving is less than the rate at which it is being processed.
Therefore, the amount of unprocessed gravel at the plant is decreasing at time $t = 5$.

$2 : \begin{cases} 1 : \text{compares } G(5) \text{ to } 100 \\ 1 : \text{conclusion} \end{cases}$

(d) The amount of unprocessed gravel at time t is given by
$A(t) = 500 + \int_0^t (G(s) - 100)\,ds.$

$A'(t) = G(t) - 100 = 0 \Rightarrow t = 4.923480$

t	$A(t)$
0	500
4.92348	635.376123
8	525.551089

The maximum amount of unprocessed gravel at the plant during this workday is 635.376 tons.

$3 : \begin{cases} 1 : \text{considers } A'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

1. On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

$G(t) = 90 + 45\cos\frac{t^2}{18}$

$G'(t) = -5t\sin\left(\frac{t^2}{18}\right)$

$G'(5) = -24.588$ tons/hr^2

This means that the rate at which unprocessed gravel arrives at the processing plant is changing by -24.588 tons per hour per hour, or decreasing by 24.588 tons per hour per hour, at $t = 5$ hours.

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

$\int_0^8 \left[90 + 45\cos\left(\frac{t^2}{18}\right)\right]dt = 825.551$ tons

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Continue problem 1 on page 5.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

Let $V(t)$ be the amount of unprocessed gravel.
$V'(t)$ is the rate at which the amount of unprocessed gravel is changing.

$V'(t) = G(t) - 100$

$V'(5) = G(5) - 100$

$= 90 + 45\cos\left(\frac{5^2}{18}\right) - 100$

$V'(5) = -1.859$

Since $V'(5)$ is negative, the amount of unprocessed gravel is decreasing at time $t = 5$ hours

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

$V'(t) = 0$ at $t = ?$

$0 = G(t) - 100$

$100 = 90 + 45\cos\left(\frac{t^2}{18}\right)$

$10 = 45\cos\left(\frac{t^2}{18}\right)$

$t = 4.923$

$V(t) - V(0) = \int_0^t (G(x) - 100)\,dx$

$V(t) = \int_0^t (G(x) - 100)\,dx + V(0)$

$V(0) = 500$

$V(4.923) = 635.376$

$V(8) = 525.551$

Since $V(t)$ is on a closed interval $[0, 8]$, the maximum amount must occur at an endpoint or at a critical value. After evaluating the amount of unprocessed gravel at $t = 0$, $t = 4.923$, and $t = 8$, the amount of unprocessed gravel is highest at $t = 4.923$, with 635.376 tons of unprocessed gravel.

GO ON TO THE NEXT PAGE.

1. On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using ~~correct units,~~ interpret your answer in the context of the problem.

$G'(5) = -24.588$ tons/hour

$G'(5)$ represents the rate of change in tons/hour² of the rate at which unprocessed gravel arrives

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

$\int_0^8 90 + 45\cos\left(\frac{t^2}{18}\right) dt$

$= 825.551$ tons

Continue problem 1 on page 5.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

rate gravel is arriving $= G(5) = 98.141$ tons/hour

rate gravel is being processed $= 100$ tons/hour

The amount of unprocessed gravel at the plant at time $t=5$ hours is decreasing since the rate at which the gravel is being processed is greater than the rate at which it is arriving.

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

$500 + \int_0^t 90 + 45\cos\left(\frac{t^2}{18}\right)$

$90 + 45\cos\left(\frac{t^2}{18}\right) = 100$

$t = 4.923$

$500 + \int_0^{4.923} 90 + 45\cos\left(\frac{t^2}{18}\right) dt$

1127.676 tons

this occurs at the time when the rate of the amount arriving equals the rate it is being processed

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1. On a certain workday, the rate, in tons per hour, at which unprocessed gravel arrives at a gravel processing plant is modeled by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, where t is measured in hours and $0 \le t \le 8$. At the beginning of the workday $(t = 0)$, the plant has 500 tons of unprocessed gravel. During the hours of operation, $0 \le t \le 8$, the plant processes gravel at a constant rate of 100 tons per hour.

(a) Find $G'(5)$. Using correct units, interpret your answer in the context of the problem.

$$G'(t) = -5t\sin\left(\frac{t^2}{18}\right)$$

$$G'(5) = -5(5)\sin\left(\frac{25}{18}\right)$$

$$G'(5) = -25\sin\left(\frac{25}{18}\right) \approx \boxed{-24.588 \text{ tons per hour}^2}$$

(b) Find the total amount of unprocessed gravel that arrives at the plant during the hours of operation on this workday.

$$\int_0^8 \left(90 + 45\cos\left(\frac{t^2}{18}\right)\right)dt = \boxed{825.551 \text{ tons}}$$

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Continue problem 1 on page 5.

(c) Is the amount of unprocessed gravel at the plant increasing or decreasing at time $t = 5$ hours? Show the work that leads to your answer.

$G(5) = 90 + 45\cos\left(\frac{5^2}{18}\right)$

$G(5) = 98.1408$

positive amount arriving

increasing

(d) What is the maximum amount of unprocessed gravel at the plant during the hours of operation on this workday? Justify your answer.

at $t=8$, max amount

at end of day

~~$G(8) = 90 + 45\cos\left(\frac{8^2}{18}\right)$~~

$\int 90 + 45\cos\left(\frac{t^2}{18}\right)dt = 825.551$ tons

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Question 1

Overview

This problem provided information related to the amount of gravel at a gravel processing plant during an eight-hour period. The function G, given by $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$, models the rate, in tons per hour, at which gravel arrives at the plant. The problem also stated that gravel is processed at a constant rate of 100 tons per hour. In part (a) students were asked to find $G'(5)$, the derivative of G at time $t = 5$. This value is negative, so students should have interpreted the absolute value of this number as the rate at which the rate of arrival of gravel at the plant is decreasing, in tons per hour per hour, at time $t = 5$. In part (b) students were asked to find the total amount of unprocessed gravel arriving at the plant over the eight-hour workday. Students should have evaluated the definite integral $\int_0^8 G(x)\,dx$, recognizing that integrating the rate at which gravel arrives over a time interval gives the net amount of gravel that arrived over that time interval. Part (c) asked whether the amount of unprocessed gravel at the plant is increasing or decreasing at time $t = 5$. Students determined whether the rate at which unprocessed gravel is arriving is greater than the rate at which gravel is being processed, i.e., whether $G(5) > 100$. Part (d) asked students to determine the maximum amount of unprocessed gravel at the plant during this workday. Because the amount of unprocessed gravel at the plant at time t is given by $A(t) = 500 + \int_0^t (G(s) - 100)\,ds$, students needed to identify the critical points of this function (where $G(t) = 100$) and to determine the global maximum on the interval $[0, 8]$. This could have been done by observing that there is a unique critical point on the interval, which is a maximum, and determining the amount of unprocessed gravel at the plant at that time, or by computing the amount of unprocessed gravel at this critical point and at the endpoints for comparison.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 1 point in part (a), 2 points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student presents a correct value for $G'(5)$ and earned the first point. The student does not address time $t = 5$ in the interpretation of the value, so the second point was not earned. In parts (b) and (c), the student's work is correct. In part (d) the first point was earned for considering where $G(t) = 100$. The student does not correctly determine the maximum amount of gravel, so the second point was not earned. A justification for a global maximum was not provided, so the third point was not earned.

Sample: 1C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (b). In part (a) the student presents a correct value for $G'(5)$ and earned the first point. The student does not provide an interpretation of this value, so the second point was not earned. In part (b) the student's work is correct. In part (c) the student ignores the rate at

Question 1 (continued)

which gravel was being processed and did not earn either point. In part (d) the student again does not consider the rate at which gravel was being processed. The student did not earn any points in this part.

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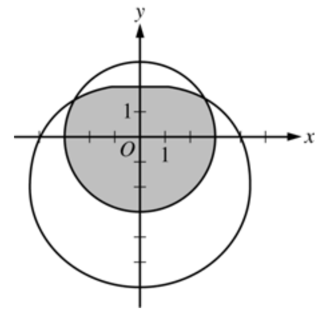
Question 2



The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

(a) Area $= 6\pi + \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4 - 2\sin\theta)^2 \, d\theta = 24.709$ (or 24.708)

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \\ 1 : \text{answer} \end{cases}$

(b) $x = r\cos\theta \Rightarrow x(\theta) = (4 - 2\sin\theta)\cos\theta$

$x(t) = \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right)$

$x(t) = -1$ when $t = 1.428$ (or 1.427)

$3 : \begin{cases} 1 : x(\theta) \text{ or } x(t) \\ 1 : x(\theta) = -1 \text{ or } x(t) = -1 \\ 1 : \text{answer} \end{cases}$

(c) $y = r\sin\theta \Rightarrow y(\theta) = (4 - 2\sin\theta)\sin\theta$

$y(t) = \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right)$

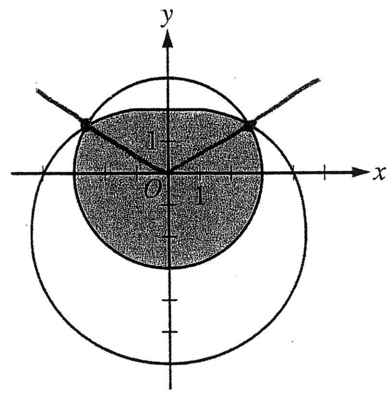
Position vector $= \langle x(t), y(t) \rangle$

$= \left\langle \left(4 - 2\sin\left(t^2\right)\right)\cos\left(t^2\right), \left(4 - 2\sin\left(t^2\right)\right)\sin\left(t^2\right) \right\rangle$

$v(1.5) = \langle x'(1.5), y'(1.5) \rangle$

$= \langle -8.072, -1.673 \rangle$ (or $\langle -8.072, -1.672 \rangle$)

$3 : \begin{cases} 2 : \text{position vector} \\ 1 : \text{velocity vector} \end{cases}$



2. The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

$$S = \frac{2}{3}\pi(3)^2 + \frac{1}{2}\int_{\pi/6}^{5\pi/6}(4-2\sin\theta)^2 = 24.709$$

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Continue problem 2 on page 7.

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(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

$r = 4 - 2\sin(t^2)$

$x = r\cos\theta$

$x = (4 - 2\sin(t^2))(\cos(t^2))$

$-1 = 4\cos t^2 - 2\sin t^2 \cos t^2$

$\boxed{t = 1.428}$ ← only t in interval $[1,2]$

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

$\langle x(t), y(t) \rangle$

$x(t) = (4\cos t^2 - 2\sin t^2 \cos t^2)$

$y = r\sin\theta$

$y = (4 - 2\sin t^2)(\sin t^2)$

$y = 4\sin t^2 - 2\sin^2 t^2$

Position vector:

$\langle 4\cos t^2 - 2\sin t^2 \cos t^2, 4\sin t^2 - 2\sin^2 t^2 \rangle$

$\left.\frac{dx}{dt}\right|_{t=1.5} = -8.0721$

Velocity vector at $t = 1.5$

$\langle -8.072, -1.673 \rangle$

$\left.\frac{dy}{dt}\right|_{t=1.5} = -1.6729$

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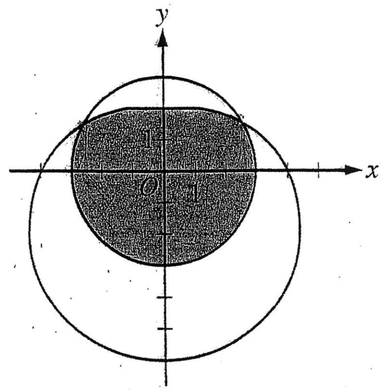

2. The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

$S = \pi(3)^2 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4-2\sin\theta)^2 =$

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Continue problem 2 on page 7.

(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

$x = r\cos\theta$

$x = (4-2\sin\theta)(\cos\theta)$

$-1 = (4-2\sin(t^2))(\cos(t^2))$

$t = 1.380$

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

position: $\langle [(4-2\sin(t^2))(\cos(t^2))], [(4-2\sin(t^2))(\sin(t^2))] \rangle$

velocity at $t=1.5$:

$\langle 1.265, 5.865 \rangle$

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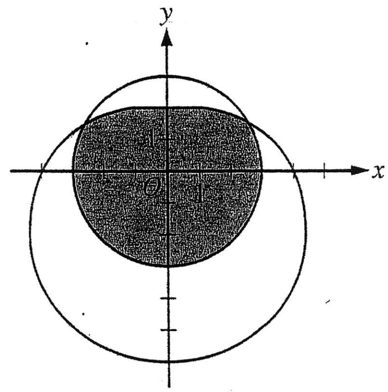

2. The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

(a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

$$\int_{\pi/6}^{5\pi/6} 3^2 - (4-2\sin\theta)^2 \, d\theta = 7.131$$

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Continue problem 2 on page 7.

(b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

~~$r = 4 - 2\sin t^2$~~

~~$-1 = 4 - 2\sin t^2$~~

~~$-3 = -2\sin t^2$~~

~~$\frac{3}{2} = \sin t^2$~~

$x = r\cos\theta$

$x = 4 - 2\sin\theta\,(\cos\theta)$

$-1 = 4 - 2\sin(t^2)(\cos t^2)$

$-3 = -2\sin t^2 \cos t^2$

$\frac{3}{2} = \sin t^2 \cos t^2$

$-1 = r\cos\theta$

(c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

$x = r\cos\theta$

$y = r\sin\theta$

$(r\cos t^2, r\sin t^2)$

$((4 - 2\sin t^2)\cos t^2, (4 - 2\sin t^2)\sin t^2)$

$((4 - 2\sin(1.5)^2)\cos(1.5)^2, (4 - 2\sin(1.5)^2\sin(1.5)^2)$

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Question 2

Overview

This problem provided the graphs of two curves defined by polar equations, along with values of θ at which the curves intersect. Part (a) asked students to find the area of the region common to the interiors of both graphs. This required students to divide the region into two subregions, bounded by arcs determined by the given values of θ, and then to apply the formula for polar area on each subregion to find the total area of the region. Part (b) described a particle moving along one of the curves and asked students to find the time when the x-coordinate of the particle is -1. Students needed to express the x-coordinate of the particle in terms of the angle θ, express θ in terms of time t, and, setting the resulting expression for x in terms of t equal to -1, solve for t in the desired time interval. Part (c) asked students to find the position vector of the particle in terms of time t, which required also determining an expression for y in terms of t, and then to find the velocity vector at a given time. This final step required finding the numerical derivative of each expression in the position vector at the given time.

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student earned 2 points for providing the integrand and the correct limits and constant. In part (b) the student earned 2 points for writing $x(\theta)$ and setting $x(t)$ equal to -1. In part (c) the student earned 2 points for the x- and y-components of the position vector.

Sample: 2C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (c). In part (a) the student earned 1 point for the integrand. In part (b) the student does not include parentheses in the expression $4 - 2\sin\theta$, and continues to attempt a solution with an incorrect equation. In part (c) the student earned 2 points for the x- and y-components of the position vector.

Question 3

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

(a) $C'(3.5) \approx \dfrac{C(4) - C(3)}{4 - 3} = \dfrac{12.8 - 11.2}{1} = 1.6$ ounces/min

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{units} \end{cases}$

(b) C is differentiable $\Rightarrow$ C is continuous (on the closed interval)

$\dfrac{C(4) - C(2)}{4 - 2} = \dfrac{12.8 - 8.8}{2} = 2$

Therefore, by the Mean Value Theorem, there is at least one time t, $2 < t < 4$, for which $C'(t) = 2$.

$2 : \begin{cases} 1 : \dfrac{C(4) - C(2)}{4 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(c)
$$\begin{aligned} \frac{1}{6}\int_0^6 C(t)\,dt &\approx \frac{1}{6}[2 \cdot C(1) + 2 \cdot C(3) + 2 \cdot C(5)] \\ &= \frac{1}{6}(2 \cdot 5.3 + 2 \cdot 11.2 + 2 \cdot 13.8) \\ &= \frac{1}{6}(60.6) = 10.1 \text{ ounces} \end{aligned}$$

$\frac{1}{6}\int_0^6 C(t)\,dt$ is the average amount of coffee in the cup, in ounces, over the time interval $0 \le t \le 6$ minutes.

$3 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{approximation} \\ 1 : \text{interpretation} \end{cases}$

(d) $B'(t) = -16(-0.4)e^{-0.4t} = 6.4e^{-0.4t}$

$B'(5) = 6.4e^{-0.4(5)} = \dfrac{6.4}{e^2}$ ounces/min

$2 : \begin{cases} 1 : B'(t) \\ 1 : B'(5) \end{cases}$

3A1 3A1

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

$$C'(3.5) \approx \frac{C(4)-C(3)}{4-3} = \frac{12.8-11.2}{1} = 1.6 \ \frac{oz}{min}$$

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

since $C(t)$ is differentiable for all values in $[2,4]$, we can say that there must be some value of t in $(2,4)$, such as $t=a$, such that $C'(a) = \frac{C(4)-C(2)}{4-2} = \frac{12.8-8.8}{2} = 2$

by the mean value theorem

So, yes!

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Continue problem 3 on page 15.

3A2 **NO CALCULATOR ALLOWED** 3A2

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

$$\frac{1}{6}\int_0^6 C(t)\,dt \approx \frac{1}{6} \cdot 2 \cdot (5.3 + 11.2 + 13.8)$$

$$\approx 10.1 \text{ oz}$$

this is the average value, in oz, of the amount of coffee in the cup over the interval $0 \leq t \leq 6$

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

$$B'(t) = \frac{32}{5}e^{-0.4t}$$

$$B'(5) = \frac{32}{5}e^{-2}$$

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3B1 3B1

NO CALCULATOR ALLOWED

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

$$\frac{f(4)+f(3)}{2} = \frac{12.8+11.2}{2} = \boxed{12 \text{ oz}}$$

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

Yes, according to the Mean Value Theorem.

$$c'(t) = \frac{f(b)-f(a)}{b-a}$$

$$\boxed{c'(t) = \frac{12.8-8.8}{4-2} = \frac{4}{2} = 2}$$

Continue problem 3 on page 15.

3B2 3B2

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(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

stepsize of 2

$(5.3)2 + (11.2)2 + (13.8)2$

$10.6 + 22.4 + 27.6 = 60.6$

$\frac{1}{6}\int_0^6 C(t)\,dt \approx \frac{1}{6}(60.6) = 10.1\text{oz}$

Over the course of six hours, the coffeemaker's average production is 10.1 oz per hour.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

$16 \times .4$

$B'(t) = 6.4e^{-0.4t}$

$\boxed{B'(5) = 6.4/e^2}$

GO ON TO THE NEXT PAGE.

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$C'(3.5) = \frac{12.8-11.2}{4-3} = \frac{1.6}{1} = \boxed{1.6 = C'(3.5)}$$

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

$$C'(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{14.5-8.8}{6-2} = \frac{5.7}{4} \not< 2$$

$$\begin{array}{r} 14.7 \\ -9.0 \\ \hline \end{array}$$

No because $\frac{5.7}{4}$ is less than 2

Continue problem 3 on page 15.

C₂

NO CALCULATOR ALLOWED

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

$\Delta x \cdot h + \Delta x \cdot h + \Delta x \cdot h$

$2 \cdot 5.3 + 2 \cdot 11.2 + 2 \cdot 13.8$

the meaning of $\frac{1}{6}\int^6 C(t)dt$ is the amount of ounces of water that was poured into the cup over the course of 0 seconds to 6 seconds.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

$\frac{2}{5} = .4$

$B(t) = 16 - 16e^{-.4t}$

$B'(t) = -16 \cdot -.4e^{-.4t}$

$B'(t) = \frac{+32}{5} e^{-.4(t)}$ $-\frac{2}{5} \cdot 5$

$B'(5) = \frac{32}{5} e^{-.4(5)}$

$B'(5) = \frac{32}{5} e^{-1}$

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Question 3

Overview

In this problem, a table was provided giving values of a differentiable function C at selected times between $t = 0$ and $t = 6$ minutes, where $C(t)$ represented the amount of coffee, in ounces, in a cup at time t. Part (a) asked students to approximate the derivative of the function C at $t = 3.5$ and to indicate units of measure. Because $t = 3.5$ fell between values of t given in the table, students should have constructed a difference quotient using the temperature values across the smallest time interval containing $t = 3.5$ that is supported by the table. Students should have recognized this derivative as the rate at which the amount of coffee in the cup is increasing, in ounces per minute, at time $t = 3.5$. Part (b) asked students whether there is a time t, $2 \le t \le 4$, at which $C'(t) = 2$. Students should have recognized that the hypotheses for the Mean Value Theorem hold because C is differentiable and then applied the theorem to the function on the interval $[2, 4]$ to conclude that there must be such a time t. Part (c) asked for an interpretation of $\frac{1}{6}\int_0^6 C(t)\,dt$ and a numeric approximation to this expression using a midpoint sum with three subintervals of equal length as indicated by the data in the table. Students should have recognized this expression as providing the average amount, in ounces, of coffee in the cup over the 6-minute time period. Students needed to use the values in the table at times $t = 1$, $t = 3$, and $t = 5$, with interval length 2, to compute this value. In part (d) students were given a symbolic expression for a function B that modeled the amount of coffee in the cup on the interval $0 \le t \le 6$. Students were asked to use this model to determine the rate at which the amount of coffee in the cup is changing when time $t = 5$. This was answered by computing the value $B'(5)$.

Sample: 3A
Score: 9

The student earned all 9 points.

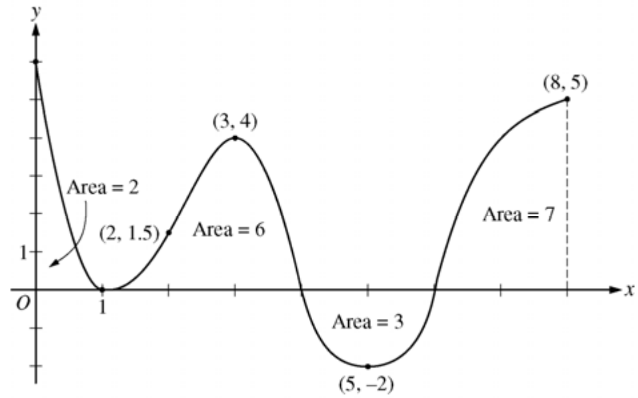
Sample: 3B
Score: 6

The student earned 6 points: 2 points in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student does not construct a numeric difference quotient, so no points were earned. The use of f instead of C is ignored. In part (b) the student's work is correct. The use of f instead of C is ignored because the student correctly uses C-values from the table and invokes the Mean Value Theorem. In part (c) the student earned the first 2 points. The interpretation point was not earned because the student does not commit to ounces as the units. The student also uses "ounces per hour" in the interpretation. In part (d) the student's work is correct.

Sample: 3C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (c), and 1 point in part (d). In part (a) the student earned the approximation point, but not the units point. In part (b) the student's work is incorrect. In part (c) the student earned the midpoint sum point, but not the approximation point. The interpretation does not include "average," so the third point was not earned. In part (d) the student has the correct derivative but incorrectly multiplies $-0.4(5)$, so the answer point was not earned.

Question 4



Graph of f'

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

(a) $x = 6$ is the only critical point at which f' changes sign from negative to positive. Therefore, f has a local minimum at $x = 6$.

1 : answer with justification

(b) From part (a), the absolute minimum occurs either at $x = 6$ or at an endpoint.

$$f(0) = f(8) + \int_8^0 f'(x)\,dx$$
$$= f(8) - \int_0^8 f'(x)\,dx = 4 - 12 = -8$$
$$f(6) = f(8) + \int_8^6 f'(x)\,dx$$
$$= f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$$
$$f(8) = 4$$

The absolute minimum value of f on the closed interval $[0, 8]$ is -8.

$$3 : \begin{cases} 1 : \text{considers } x = 0 \text{ and } x = 6 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$$

(c) The graph of f is concave down and increasing on $0 < x < 1$ and $3 < x < 4$, because f' is decreasing and positive on these intervals.

$$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$$

(d) $g'(x) = 3[f(x)]^2 \cdot f'(x)$

$$g'(3) = 3[f(3)]^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot 4 = 75$$

$$3 : \begin{cases} 2 : g'(x) \\ 1 : \text{answer} \end{cases}$$

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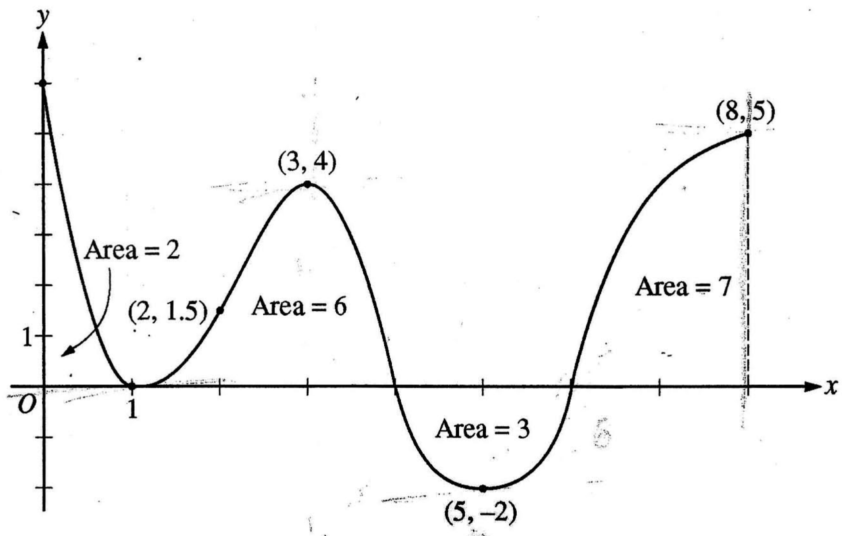

Graph of f'

4. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

$x = 6$

f has a local minimumat $x = 6$, because the graph of f ' changes from negative to positive at $X=6$, so using the first derivative test and ~~the fact the at x=6~~, f has a critical number, at $X=6$, f has a local minimum

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

local minimum = $X = 6$

$f(8) = 4$

$f(6) \rightarrow \int_6^8 f'(x)\,dx = 7 = f(8) - f(6) = 4 - f(6)$ $f(6) = -3$

$f(0) \rightarrow \int_0^8 f'(x) = 12 = f(8) - f(0) = 4 - f(0)$ $f(0) = -8$

The Absolute minimum value of f on the interval $0 \le x \le 8$ is -8 because it is the lowest value for f among the endpoints and critical numbers.

Continue problem 4 on page 17.

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

$\hookrightarrow f'' < 0$

The open intervals where the graph of f is both concave down and increasing is $(0,1) \cup (3,4)$, or $0<x<1$ and $3<x<4$, because using the graph of f', when the graph of f' is positive and the slope of f' is negative, that means that f is increasing and f'' is negative, so f is both concave down and increasing.

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

$g'(x) = 3(f(x))^2 \cdot f'(x)$

$g'(3) = 3(f(3))^2 \cdot f'(3) = 3\left(-\frac{5}{2}\right)^2 \cdot (4)$

$= 3\left(\frac{25}{\cancel{4}}\right)\cancel{4} = 75$

$g'(3) = 75$

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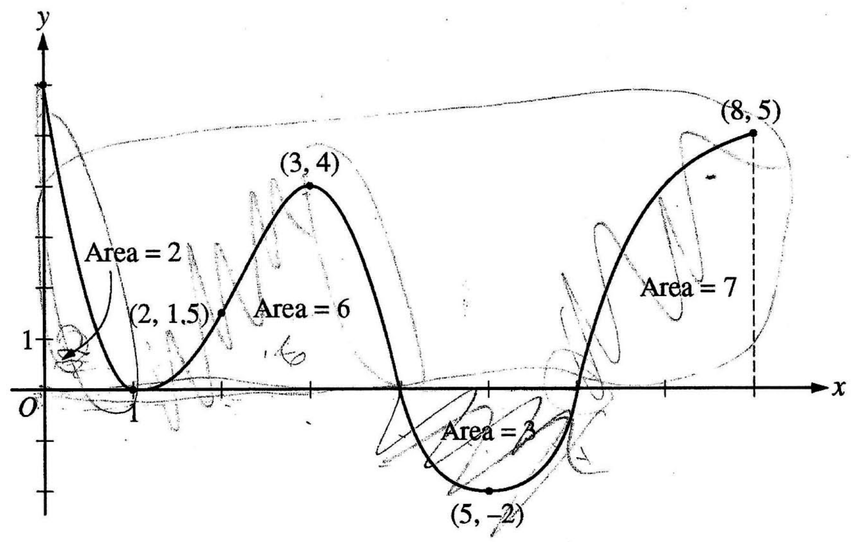

Graph of f'

4. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

$x=6$ because the sign of f' changes from negative to positive.

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

x	$\int_0^x f'(x)\,dx$
0	0
2	4
1	neither
4	relmax
6	5

$x=8$ because that is where $f(x)$ is the smallest.

Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

f(x) is concave down and increasing
when f'' is negative and f' is positive
this occurs (0,1) and (3,4).

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

$x_1 = 3$
$y_1 = -\frac{5}{2}$
$\boxed{m = -75}$

$g'(x) = 3(f(x))^2 \cdot f'(x)$
$g'(3) = 3(-\frac{5}{2})^2 \cdot 4$
$g'(3) = -\frac{75}{4} \cdot \cancel{4}$
$g'(3) = -\frac{300}{4}$
$g'(3) = -75$

$75 \times 4 = 300$

$300 \div 4 = 75$

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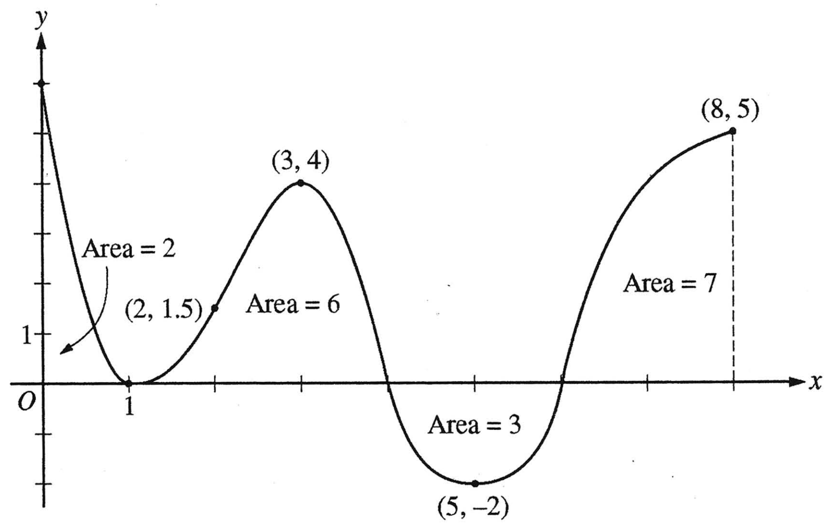

Graph of f'

4. The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the closed interval $0 \le x \le 8$. The graph of f' has horizontal tangent lines at $x = 1$, $x = 3$, and $x = 5$. The areas of the regions between the graph of f' and the x-axis are labeled in the figure. The function f is defined for all real numbers and satisfies $f(8) = 4$.

(a) Find all values of x on the open interval $0 < x < 8$ for which the function f has a local minimum. Justify your answer.

Min of f is when f' changes from (-) to (+)

Min @ X=6

(b) Determine the absolute minimum value of f on the closed interval $0 \le x \le 8$. Justify your answer.

X	$\int_0^8 f$
0	6
1	0
3	4
5	-2
8	5

$\boxed{\text{Min of -2 at X=5}}$

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Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) On what open intervals contained in $0 < x < 8$ is the graph of f both concave down and increasing? Explain your reasoning.

f is concave down when f'' is (-)

f is increasing when f' is (+)

Both from (0,1), u (3,4)

(d) The function g is defined by $g(x) = (f(x))^3$. If $f(3) = -\frac{5}{2}$, find the slope of the line tangent to the graph of g at $x = 3$.

$m = g'(x) = (f'(3))^3 = 4^3 = 48$

$x = 3$

$g(3) = (f(3))^3$

$\frac{-5^3}{3}$

$\frac{125}{9} = y$

$\frac{16}{3}$
48

$\boxed{y - \frac{125}{9} = 48(x-3)}$

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Question 4

Overview

This problem described a function f that is defined and twice differentiable for all real numbers, and for which $f(8) = 4.$ The graph of $y = f'(x)$ on $[0, 8]$ is given, along with information about locations of horizontal tangent lines for the graph of f' and the areas of the regions between the graph of f' and the x-axis over this interval. Part (a) asked for all values of x in the interval $(0, 8)$ at which f has a local minimum. Students needed to recognize that this occurs where f' changes sign from negative to positive. Part (b) asked for the absolute minimum value of f on the interval $[0, 8].$ Students needed to use the information about the areas provided with the graph, as well as $f(8),$ to evaluate $f(x)$ at 0 and at the local minimum found in part (a). Part (c) asked for the open intervals on which the graph of f is both concave down and increasing. Students needed to recognize that this is given by intervals where the graph of f' is both decreasing and positive. Students were to determine these intervals from the graph. Part (d) introduced a new function g defined by $g(x) = (f(x))^3,$ and included that $f(3) = -\frac{5}{2}.$ Students were asked to find the slope of the line tangent to the graph of g at $x = 3.$ Students needed to recognize that this slope is given by $g'(3).$ In order to determine this value, students needed to apply the chain rule correctly and read the value of $f'(3)$ from the graph.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student's work is correct. In part (b) the student earned the point for considering $x = 0$ and $x = 6.$ The student does not report a correct answer and was not eligible for the justification point. In part (c) the student's work is correct. In part (d) the student earned the 2 points for $g'(x)$ but did not earn the answer point.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a) and 2 points in part (c). In part (a) the student's work is correct. In part (b) the student does not consider $x = 0$ and $x = 6,$ does not find the answer, and is not eligible for the justification point. In part (c) the student's work is correct. In part (d) the student makes a chain rule error in the derivative and did not earn the answer point.

Question 5

Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) $\lim_{x\to 0}(f(x)+1) = -1+1 = 0$ and $\lim_{x\to 0}\sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x\to 0}\frac{f(x)+1}{\sin x} = \lim_{x\to 0}\frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2\cdot 2}{1} = 2$$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(b)
$$f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right) = -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2\cdot\frac{1}{4}+2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c)
$$\frac{dy}{dx} = y^2(2x+2)$$
$$\frac{dy}{y^2} = (2x+2)\,dx$$
$$\int\frac{dy}{y^2} = \int(2x+2)\,dx$$
$$-\frac{1}{y} = x^2+2x+C$$
$$-\frac{1}{-1} = 0^2+2\cdot 0+C \Rightarrow C = 1$$
$$-\frac{1}{y} = x^2+2x+1$$
$$y = -\frac{1}{x^2+2x+1} = -\frac{1}{(x+1)^2}$$

Note: This solution is valid for $x > -1$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

5A1

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x \to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

a) $\lim_{x\to 0} \frac{f(x)+1}{\sin x} = \frac{-1+1}{0} = \frac{0}{0}$, Using L'Hôpital, $\lim_{x\to 0}\left(\frac{f(x)+1}{\sin x}\right) = \lim_{x\to 0} \frac{\frac{d}{dx}(f(x)+1)}{\frac{d}{dx}\sin x}$

indeterminate

$\lim_{x\to 0} \frac{f'(x)}{\cos x} = \lim_{x\to 0} \frac{y^2(2x+2)}{\cos x} = \frac{(-1)^2(2)}{1} = \boxed{2}$

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

$f(1/4) \approx f(0) + \frac{1}{4}f'(0) = -1 + \frac{1}{4}(2) = -1/2$

$f(1/2) \approx f(1/4) + \frac{1}{4}f'(1/4) = -1/2 + 1/4(1/4(1/2+2))$

$$\boxed{f(1/2) \approx -1/2 + \frac{1}{4}\left(\frac{1}{4}(1/2+2)\right)}$$

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Continue problem 5 on page 19.

5A2

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$\frac{dy}{dx} = y^2(2x+2)$, $\frac{dy}{y^2} = (2x+2)dx$

$\int \frac{1}{y^2} dy = \int (2x+2)dx$, $-1/y = x^2 + 2x + C$

$-1/_{-1} = 0^2 + 2(0) + C$, $C = 1$

$-1/y = x^2 + 2x + 1$

$$y = -\frac{1}{(x+1)^2}$$

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5B 1

NO CALCULATOR ALLOWED

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x \to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\lim_{x \to 0} \frac{f(x)+1}{\sin(x)} = f'(\sin(x))$$

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

x	y	Δx	$\frac{dy}{dx}$ $y^2(2x+2)$	$\Delta y = \frac{dy}{dx} \cdot \Delta x$	$x + \Delta x$ new x	$y + \Delta y$ new y
0	-1	$\frac{1}{4}$	2	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$
$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{5}{32}$	$\frac{1}{2}$	$-\frac{11}{32}$

$f\left(\frac{1}{2}\right) \approx -\frac{11}{32}$

$$\left(-\frac{1}{2}\right)^2\left(2 \cdot \frac{1}{4} + 2\right)$$

$$\frac{1}{2} + 2 =$$

$$\frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$$

$$-\frac{1}{2} + \frac{5}{32} =$$

$$-\frac{16}{32} + \frac{5}{32} = -\frac{11}{32}$$

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Continue problem 5 on page 19.

5B2

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(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$$\frac{dy}{dx} = y^2(2x+2)$$

$$\frac{dy}{y^2} = (2x+2)\,dx$$

$$\frac{-1}{y} = x^2 + 2x + C$$

$$\frac{-1}{-1} = \cancel{0}^2 + 2 0 + C \qquad C = 1$$

$$y = x^2 + 2x + 1$$

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5C₁

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

from part (c)

$$\boxed{1 - e^{-\frac{1}{2}}}$$

$$\lim_{x\to 0} \frac{\left(\sqrt{e^{(x^2+2x)}} - \frac{1}{\sqrt{e}}\right)+1}{\sin x}$$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{0}{0}$ or und

$$\lim_{x\to 0} \frac{\left(e^{(x^2+2x)}\right)^{\frac{1}{2}} - e^{-\frac{1}{2}} + 1}{\sin x}$$

$\lim_{x\to 0} \frac{f'(x)}{g'(x)}$

L'Hopital rule

$$\lim_{x\to 0} = \frac{\frac{1}{2}(2x+2)\left(e^{(x^2+2x)}\right)^{-\frac{1}{2}} - e^{-\frac{1}{2}}}{\cos(x)} = \frac{\frac{2(1)}{2} - e^{-\frac{1}{2}}}{1}$$

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

$(0, -1)$

$\frac{dy}{dx} = y^2(2x+2)$

$y_n = y_{old} + f(x_{old}, y_{old}) \cdot \Delta x$

$\left(\frac{1}{4}\right)\left(\frac{5}{2}\right)$

$\frac{5}{8}$

$f(.25) = -1 + (2) \times (0.25)$

$f(.25) = -\frac{1}{2}$ $\quad (0.25, -\frac{1}{2})$

$f\left(\frac{1}{2}\right) = -\frac{1}{2} + \left(\frac{5}{8}\right) \times \left(\frac{1}{4}\right)$

$$\boxed{f\left(\frac{1}{2}\right) = -\frac{11}{32}}$$

$f\left(\frac{1}{2}\right) = -\frac{1}{2} + \frac{5}{32}$ $\quad f\left(\frac{1}{2}\right) = \frac{5}{32} - \frac{16}{32}$

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Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

5C2

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$$\left(\frac{dy}{dx} = \frac{y^2(2x+2)}{y^2}\right) dx$$

$$\int \frac{1}{y^2}\,dy = \int (2x+2)\,dx$$

$$\ln(y^2) = \frac{2x^2}{2} + 2x$$

$$\ln(y^2) = x^2 + 2x$$

$$\sqrt{y^2} = \sqrt{e^{x^2+2x}}$$

$$y = \sqrt{e^{(x^2+2x)}} + c$$

$$0 = \sqrt{e^{(-1)^2+2(-1)}} + c$$

$$0 = \sqrt{e^{1-2}} + c$$

$$0 = \sqrt{e^{-1}} + c$$

$$\boxed{f(x) = \sqrt{e^{(x^2+2x)}} - \frac{1}{\sqrt{e}}}$$

$$0 = \frac{1}{\sqrt{e}} + c$$

$$\boxed{c = -\frac{1}{\sqrt{e}}}$$

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Question 5

Overview

This problem presented students with a differential equation and defined $y = f(x)$ to be the particular solution to the differential equation satisfying a given initial condition. Part (a) asked students to compute a limit of an expression involving $f(x)$, which required students to apply L'Hospital's Rule. Part (b) asked students to use Euler's method with two steps of equal size to approximate $f(x)$ at a value near the point given by the initial condition. Part (c) asked for the particular solution to the differential equation satisfying the given initial condition. Students should have used the method of separation of variables to solve the differential equation.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (b) and 4 points in part (c). The work in part (a) will not lead to an application of L'Hospital's Rule. In part (b) the student's work is correct. In part (c) the student has a correct separation of variables and the correct antiderivatives. The student correctly places the constant of integration and uses the initial condition. The student does not correctly solve for y and did not earn the fifth point.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (b) and 1 point in part (c). In part (a) the student describes the conditions for using L'Hospital's Rule. The student imports an incorrect function from part (c). This function does not produce an indeterminate form; therefore, the application of L'Hospital's Rule is incorrect. The student did not earn any points in part (a). In part (b) the student's work is correct. In part (c) the student shows the separation of variables. The student incorrectly uses a logarithm when taking the antiderivative of $\frac{1}{y^2}$. The student did not earn the antiderivatives point or any of the remaining points.

Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$

$f'(0) \cdot \frac{1}{2} = 1$

$f'(0) = 2$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$

$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$

$Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$

$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

OR

$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$

$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$

$Q_3(x) = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$

6A1

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$$P_n = f(c) + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!}$$

$$P_1 = -4 + f'(0)x$$

$$-3 = -4 + f'(0) \cdot \tfrac{1}{2}$$

$$1 = \frac{f'(0)}{2}, \therefore f'(0) = 2$$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$$P_3(x) = -4 + 2x + \frac{(-2/3)x^2}{2!} + \frac{(1/3)x^3}{3!}$$

$$\boxed{= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3}$$

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Continue problem 6 on page 21.

6A2

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(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$h(x) = \int f(2x)\,dx \approx \int P_3(2x)$ $\quad \frac{4x^2}{3} \quad \frac{4}{9}x^3$

$P_3(2x) = -4 + 2(2x) - \frac{1}{3}(2x)^2 + \frac{1}{18}(2x)^3$

$h(0) + \int P_3(2x) = -4x + 2x^2 - \frac{4}{9}x^3$

$\boxed{P_3 \text{ for } h = 7 - 4x + 2x^2 - \frac{4}{9}x^3}$

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6B₁

NO CALCULATOR ALLOWED

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$$P_1(x) = -4 + f'(0)\left(\tfrac{1}{2}\right) = -3$$

$$f'(0)\left(\tfrac{1}{2}\right) = 1$$

$$f'(0) = 2$$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$$P_3(x) = \boxed{-4 + 2x - \frac{2/3\, x^2}{2} + \frac{1/3\, x^3}{6}}$$

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Continue problem 6 on page 21.

6B2

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$$\cancel{T_3(x) = 7 + 2(2x) - \frac{2/3(2x)^2}{2} + \frac{1/3(2x)^2}{6}}$$

$$\cancel{\boxed{= 7 \quad 4x - \frac{4}{3}x^2 + \frac{2x^3}{9}}}$$

$$T_3(x) = 7 - 4(2x) + \frac{2(2x)^2}{2} - \frac{2/3(2x)^3}{6}$$

$$\frac{8 \cdot 2}{3 \cdot 6}$$

$$\boxed{= 7 - 8x + 4x^2 - \frac{8}{9}x^3}$$

GO ON TO THE NEXT PAGE.

6C1

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$-4 + x = -3$

$x = 1$

$f'(0) = 2$

$-4 + \frac{2x}{1!} = -3$

$2x = 1$

$x = \frac{1}{2}$

$P_1(\frac{1}{2}) = -3$

$\therefore\ f'(0) = 2$

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$f(0) = -4$

$f'(0) = 2$

$f''(0) = -2/3$

$f'''(0) = 1/3$

$-4 + \frac{2x^2}{2!} - \frac{2x^3}{3 \cdot 3!} + \frac{x^4}{3 \cdot 4!}$

$\boxed{-4 + x^2 - \frac{x^3}{9} + \frac{x^4}{72}}$

$\frac{24}{3}$

72

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Continue problem 6 on page 21.

6C2

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(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

$h(0) = 7$ $\quad h'(0) = f(0)$

$h'(x) = f(2x)$ $\quad h''(0) = f'(2\cancel{x})(2)$

$7 + \frac{-4x}{1!} + \frac{4x^2}{2!}$

$\boxed{7 - 4x + 2x^2}$

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Question 6

Overview

This problem described a function f known to have derivatives of all orders at $x = 0$. In part (a) information was provided about the value of the function at $x = 0$, as well as about the value of its first-degree Taylor polynomial about $x = 0$ at the point $x = \frac{1}{2}$. Students needed to use this information to verify that $f'(0) = 2$. In part (b) students were given information about the second and third derivatives of f at $x = 0$. Students needed to use this additional information to find the third-degree Taylor polynomial for f about $x = 0$. In part (c) a new function h was defined in terms of f. Students needed to use information provided about $h(0)$, as well as information already provided about f, to find the third-degree Taylor polynomial for h about $x = 0$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In parts (a) and (b), the student's work is correct. No supporting work was required. In part (c) the student appears to have antidifferentiated $P_2(x)$ and replaced x by $2x$ in the result. This is not a legitimate method, and the student did not earn the fourth point. The student earned the third point because the answer has a constant term of 7 in a polynomial of degree three or higher.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a) and 1 point in part (c). In part (a) the student's actual solution begins on the third line and earned both points in part (a). The student shows that $f'(0) = 2$ together with the given values for $P_1\left(\frac{1}{2}\right)$ and $f(0)$ implies that $x = \frac{1}{2}$, thereby verifying that $f'(0) = 2$. In part (b) the student's work is incorrect. In part (c) the student earned 1 of 2 points. The student would have earned the second point with a correct expression for $h'''(0)$. Because the student attempts to use a legitimate method, the student was eligible to earn the fourth point. No additional points were earned since the answer is not a polynomial of degree three or higher, and the student does not have a value for $h'''(0)$.